Conformal Transformation and it's Applications

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Abstract
This paper investigates flow past a flat plate and two-dimensional irrotational motion of fluid due to singularities between two fixed boundaries. The flow past a flat plate is obtained by using Joukowski transformation. It is also shown by examples that the conformal transformation can make a problem of irrotational flow treatable by converting an awkwardly shaped boundary into one of the simple forms.
Keywords: flat plate, Joukowski transformation, conformal transformation, singularities.

1. Conformal Transformation
Suppose that z and ζ are two complex variables defined by \( z = x + iy \) and \( ζ = ξ + iη \) where \( x, y, ξ, η \) are real variables. Suppose that \( z \) describes a certain curve \( C \) in the \( z \)-plane and \( ζ \) is related to \( z \) by means of the transformation \( ζ = f(z) \) where \( f(z) \) is analytic.

If \( f(z) \) is a single-valued function of \( z \), then to each point in the \( z \)-plane, we can obtain a corresponding point in the \( ζ \)-plane. In this way, the curve \( C \) in the \( z \)-plane may be mapped into a curve \( C' \) in the \( ζ \)-plane.

\[ \begin{align*}
\text{(z-plane)} & \quad \text{(ζ-plane)} \\
C & \quad f(C) \\
\zeta_0 & \quad f(\zeta_0) \\
\zeta_1 & \quad f(\zeta_1)
\end{align*} \]

Figure 1

Suppose that the function \( f(z) \) is analytic. Let \( P, Q, R \) be neighboring points in the \( z \)-plane such that \( OP = z, OQ = z + \delta z_1, OR = z + \delta z_2 \).

\[ \begin{align*}
\text{(z-plane)} & \quad \text{(ζ-plane)} \\
\begin{array}{c}
\text{O} \\
\text{P(z)} \\
\text{Q(z+δz_1)} \\
\text{R(z+δz_2)}
\end{array} & \quad \begin{array}{c}
\text{O} \\
\text{P'(ζ)} \\
\text{Q'(ζ+δζ_1)} \\
\text{R'(ζ+δζ_2)}
\end{array}
\end{align*} \]

Figure 2

Under the transformation \( ζ = f(z) \), suppose that \( P, Q, R \) map into the points \( P', Q', R' \) respectively in the \( ζ \)-plane, where \( OP' = ζ, OQ' = ζ + δζ_1, OR' = ζ + δζ_2 \). It is assumed that

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$|\delta z_1|, |\delta z_2|, |\delta \zeta_1|, |\delta \zeta_2|$ are small. Since $f(z)$ is analytic, $\frac{dz}{dz}$ is unique at P. Thus, to the first order of smallness,

\[
\frac{\delta \zeta_1}{\delta z_1} = \frac{\delta \zeta_2}{\delta z_2} \quad \text{(or)} \quad \frac{\delta \zeta_1}{\delta \zeta_2} = \frac{\delta z_1}{\delta z_2} .
\]

Therefore,

\[
\left| \frac{\delta \zeta_1}{\delta \zeta_2} \right| = \left| \frac{\delta z_1}{\delta z_2} \right| .
\]

(1)

\[
\delta \zeta_1 - \arg \delta \zeta_2 = \arg \delta z_1 - \arg \delta z_2 ,
\]

(3)

and

\[
\delta \zeta_1 = \frac{\delta \zeta_1}{\delta z_1} \delta z_1.
\]

Therefore, $|\delta \zeta_1| = \frac{|\delta \zeta_1|}{|\delta z_1|}$ and $\arg \delta \zeta_1 = \arg \left( \frac{\delta \zeta_1}{\delta z_1} \right) + \arg \delta z_1$. So in the neighborhood of the point $P'$ distances are multiplied by the value of $\left| \frac{\delta \zeta_1}{\delta z_1} \right|$ at $P$; this is called the magnification of the transformation.

From (2) and (3), we obtain $\frac{P'Q'}{PR'} = \frac{PQ}{PR}$ and $\angle RP'Q' = \angle RPQ$. Thus the triangles $R'P'Q'$ and $RPQ$ are similar. So an infinitesimal triangle in the $z$-plane maps into a similar infinitesimal triangle in the $\zeta$-plane. Thus the mapping preserves the angles and the similarity of corresponding infinitesimal triangles. Such a transformation which has these properties is said to be conformal.

Example: Transformation of $w = z^2$

1.1 Applications of Conformal Transformation

Suppose there is a two-dimensional incompressible flow in the $z$-plane. On applying the conformal transformation $\zeta = g(z)$, the new plane of flow becomes the $\zeta$-plane. Let $\rho$ be the density of the fluid in both cases. Suppose further that $C$ is a rigid boundary in the $z$-plane which maps into the curve $C'$ in the $\zeta$-plane. Let the complex velocity potential for the $z$-plane be $w = f(z) = \phi + i \psi$ where the real functions $\phi(x, y), \psi(x, y)$ are the usual velocity potential and stream function respectively. By means of the transformation $\zeta = g(z)$ it can
express \( w \) as a function \( \Phi(z) = \phi + i \psi \) where \( \phi = \phi(\xi, \eta), \psi = \psi(\xi, \eta) \). At the corresponding points \( t, z \), the complex potential \( w \) takes the same value so that \( \phi = \bar{\phi}, \psi = \bar{\psi} \).

Now \( C \) is a rigid boundary in the \( z \)-plane and so also a streamline for which \( \psi = \text{constant} \). Thus along \( C', \psi = \text{constant} \). Therefore, \( C' \) is a streamline and also a rigid boundary. Therefore, under the conformal transformation, points on the streamline through a given point in the \( z \)-plane will transform into points on the stream line through the corresponding point in the \( \zeta \)-plane. In particular, the boundaries of the fluid in \( z \)-plane will transform the boundaries in \( \zeta \)-plane.

### 1.2 Transformation of source and sink

![Diagram](image_url)

Figure 4

Suppose there is a source of strength \( m \) at \( P \) in the \( z \)-plane surrounded by a small closed curve \( C \). By the definition of source, the flow across \( C \) is \( 2\pi m \rho \). Under the conformal transformation the point \( P \) transforms into the point \( P' \) in the \( \zeta \)-plane and the small closed curve \( C \) surrounding \( P \) in the \( z \)-plane transforms into a small closed curve \( C' \) surrounding \( P' \) in the \( \zeta \)-plane. The flow across \( C \) is given in terms of the stream function by \( -\rho \oint_C d\psi \). Since each point on \( C' \) corresponds to one and only one point on \( C \), this is equal to \( -\rho \oint_{C'} d\psi \) taken in the same sense. So, the flow across \( C' \) is \( 2\pi m \rho \) and this will be the same for any small closed curve surrounding \( P' \). Therefore a source transforms into an equal source at the corresponding point. Similarly if there is a sink of strength \( -(\rho m) \) at \( P \) in the \( z \)-plane, then it transforms into an equal sink at the corresponding point in the \( \zeta \)-plane.

In particular, the boundaries of the fluid in \( z \)-plane will transform the boundaries in \( \zeta \)-plane. And a source, sink or vortex at a particular point in the \( z \)-plane will transform an equal source, sink or vortex at the corresponding point in the \( \zeta \)-plane. The kinetic energy of both corresponding regions is equal.

### 2. Joukowski Transformation

It is the most common conformal transformation which is given by

\[
\zeta = f(z) = z + \frac{a^2}{z},
\]

where \( a \) is constant. The transformation changes \( z \)-plane to \( \zeta \)-plane where \( z = x + iy \). Then,

\[
\zeta = \xi + i\eta = z + \frac{a^2}{z},
\]
\[ \xi = x + \frac{a^2 (x - iy)}{(x + iy)(x - iy)} \]

Therefore, \( \xi = x (1 + \frac{a^2}{x^2 + y^2}) \), \( \eta = y (1 - \frac{a^2}{x^2 + y^2}) \). \( \xi = \eta, \eta = -\xi \) \( (5) \)

If \( x^2 + y^2 = r^2 \), the circle of radius \( r \) is in the \( z \)-plane, then

\[ \frac{\xi^2}{r^2 + \frac{a^2}{r} - \frac{a^2}{r}} + \frac{\eta^2}{r^2 - \frac{a^2}{r}} = 1 \), in the \( \zeta \)-plane. 
\[ (6) \]

Therefore by using Joukowski transformation, a circle on the \( z \)-plane of radius \( r \) transforms into an ellipse with major axis \( A = r + \frac{a^2}{r} \) and minor axis \( B = r - \frac{a^2}{r} \) on the \( \zeta \)-plane.

In the special case, when \( r = a \), the ellipse becomes an infinitely thin plate of length \( 4a \) in the \( \zeta \)-plane, since \( A = 2a \) and \( B = 0 \). So, the Joukowski transformation changes the circle into a flat plate. And then the circle of radius \( a \) in the \( z \)-plane is called the Joukowski transformation circle.

2.1 Flow Past a Flat Plate with Circulation

The complex potential for a fixed circular cylinder radius \( a \) in a stream whose undisturbed speed \( U \) makes an angle \( \alpha \) with the X-axis and about which there is a circulation \( \kappa \) is

\[ W = U \left( z e^{i\alpha} + \frac{a^2 e^{-i\alpha}}{z} \right) + \frac{i\kappa}{2\pi} \log z. \] \( (7) \)
If the transformation \( \zeta = \frac{a^2}{z} + z \) is applied to the whole area outside the circle in the \( z \)-plane, it transforms into the whole of the \( \zeta \)-plane with a rigid barrier between the points \((\pm 2a, O)\). The problem then becomes that a flat plate of width \( 4a \), about which there is circulation, in a stream \( U \) inclined at \( \alpha \) to the plate. Solving for \( z \) in terms of \( \zeta \),

\[
z = \frac{1}{2} \left\{ \zeta + \sqrt{\zeta^2 - 4a^2} \right\} \quad \text{and} \quad \frac{a^2}{z} = \frac{1}{2} \left\{ \zeta - \sqrt{\zeta^2 - 4a^2} \right\}.
\]

Hence the complex potential (7) becomes

\[
W = \frac{1}{2} U \left[ \left\{ \zeta + \sqrt{\zeta^2 - 4a^2} \right\} e^{i\alpha} + \left\{ \zeta - \sqrt{\zeta^2 - 4a^2} \right\} e^{-i\alpha} \right]
\]

\[
+ \frac{ik}{2\pi} \log \frac{1}{2} \left\{ \zeta + \sqrt{\zeta^2 - 4a^2} \right\}.
\]

\[
= \frac{U}{2} \left[ \zeta \cos\alpha + i \left\{ \sqrt{\zeta^2 - 4a^2} \right\} \sin\alpha \right]
\]

\[
+ \frac{ik}{2\pi} \log \frac{1}{2} \left\{ \zeta + \sqrt{\zeta^2 - 4a^2} \right\},
\]

neglecting a constant.

The circulation about the plate is given by the decrease in the velocity potential \( \phi \) on describing a circuit round it, this is the same as the decrease in \( \phi \) on describing the corresponding circuit about the cylinder, i.e., there is a circulation \( \kappa \) about the plate.

The velocity at any point can be written

\[
-U + iV = \frac{dW}{d\zeta} = \frac{dW}{dz} \left( 1 - \frac{a^2}{z^2} \right).
\]

The denominator vanishes when \( z = \pm a \), i.e., \( \zeta = \pm 2a \), therefore the velocity is infinite at both edges unless \( \frac{dW}{dz} \) has a factor \((z + a)\) or \((z - a)\), when it will be finite at the corresponding edge.

\[
\frac{dW}{dz} = U \left( e^{i\alpha} - \frac{a^2}{z^2} e^{-i\alpha} \right) + \frac{ik}{2\pi z},
\]

if this is zero when \( z = \pm a \), then \( \kappa = \pm 4\pi U a \sin \alpha \). Hence the velocity at the edge \( \zeta = 2a \) will be finite.

**Example**

A flat plate of infinite length and width \( L \) is placed in a current of incompressible fluid with its plane at an angle \( \alpha \) to the undisturbed stream lines.
The circle of radius $a$ on BOA as diameter transforms into the flat plate $B'A'$ of length $4a$, by means of transformation $\zeta = z + \frac{a^2}{z}$. Taking the centre of the circle as origin, the complex potential in $z$-plane is given by

$$W = Uz e^{ia} + \frac{Ua^2}{z} e^{-ia} + \frac{ik}{2\pi} \log z.$$ 

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Stagnation points corresponding to $z = \pm a$ are given by $\frac{dW}{dz} = 0$. Therefore $\kappa = 4\pi a U \sin \alpha$.

By using the Blasiu's theorem, 

$$X - iY = \frac{1}{2} \rho \int \left( \frac{dW}{d\zeta} \right)^2 d\zeta.$$

$$= \frac{1}{2} \rho \int \left( 1 + \frac{a^2}{z^2} + \ldots \right) \left( U e^{ia} - \frac{Ua^2}{z^2} e^{-ia} + \frac{ik}{2\pi z} \right)^2 \, dz.$$

By using Residue Theorem, 

$$X - iY = \frac{1}{2} \rho \int \frac{2U i \kappa e^{ia}}{2\pi} - 2\pi i = -i\rho \kappa U e^{ia}$$

$$X = \rho \kappa U \sin \alpha = 4\pi a U^2 \sin^2 \alpha$$

$$Y = \rho \kappa U \cos \alpha = 4\pi a U^2 \sin \alpha \cos \alpha.$$

The resultant force $R$ is $4\pi a U^2 \sin \alpha$ and acting at angle, $\tan \theta = \frac{Y}{X}$, $\theta = \frac{\pi}{2} - \alpha$.

$$N = \text{real part of} \, -\frac{1}{2} \rho \int \left( \frac{dW}{d\zeta} \right)^2 \zeta d\zeta$$

$$= -\frac{1}{2} \rho \int \frac{z + a^2}{1 - \frac{a^2}{z^2}} \left( \frac{dW}{dz} \right)^2 dz.$$
\[ = \text{real part of} \left[ -\frac{1}{2} \rho \left( 2U^2 a^2 e^{2i\alpha} - \frac{\kappa^2}{4\pi^2} - 2U^2 a^2 \right) 2\pi i \right] \]
\[ = 2\pi \rho U^2 a^2 \sin 2\alpha = \frac{\pi}{8} \rho L^2 U^2 \sin 2\alpha. \]

3. Flow Due to a Source between Two Fixed Boundaries

Consider a source \( m \) at the point \( z_0 \) in the fluid bounded by the lines \( \theta = 0 \) and \( \theta = \frac{\pi}{3} \).

The conformal transformation \( Z = z^3 \) where \( z = re^{i\theta} \) from \( z \)-plane transform to \( Z \)-plane. The boundaries \( \theta = 0 \) and \( \theta = \frac{\pi}{3} \) in \( z \)-plane transform to \( \Theta = 0 \) and \( \Theta = \pi \) (real axis) in \( Z \)-plane.

The point \( z_0 \) in \( z \)-plane transforms to point \( Z_0 \) in \( Z \)-plane such that \( Z_0 = z_0^3 \) and the source \( m \) at \( z_0 \) transforms to a source \( m \) at \( Z_0 \). Hence the image system with respect to real axis in \( Z \)-plane consists of a source \( m \) at \( Z_0 \) and a source \( m \) at \( Z'_0 \). Therefore the complex potential for this motion is

\[ W = -m \log (Z - Z_0) - m \log (Z - Z'_0) \]
\[ = -m \log (z^3 - z_0^3) - m \log (z^3 - z_0'^3) \]
\[ \phi + i\psi = -m \log \left( (z^3 - z_0^3)(z^3 - z_0'^3) \right). \]

Example

Suppose that between two fixed boundaries \( \theta = \frac{\pi}{4} \) and \( \theta = -\frac{\pi}{4} \), there is two-dimensional liquid motion due to a source of strength \( m \) at the point \((a,0)\) and an equal sink at the point \((b,0)\).
Consider the transformation $Z = z^2$, from xy-plane to $\zeta \eta$-plane, where $z = re^{i\theta}$ and $Z = \text{Re}^{i\theta}$.

Hence the boundaries $\theta = \pm \frac{\pi}{4}$ in the z-plane transform to $\Theta = \pm \frac{\pi}{2}$, the imaginary axis of Z-plane. The points $A(a,0)$ and $B(b,0)$ transform to $A'(a^2,0)$ and $B'(b^2,0)$ respectively. Since the source transforms to an equal source at $A'$ and the sink transforms to an equal sink at $B'$, the image system with respective to imaginary axis in Z-plane consists of a source of strength $m$ at $A''(-a^2,0)$ and a sink of strength $-m$ at $B''(-b^2,0)$.

Therefore, the complex potential for this motion is

$$W = -m \log \left(Z - a^2\right) + m \log \left(Z - b^2\right) - m \log \left(Z + a^2\right) + m \log \left(Z + b^2\right)$$

By using the transformation,

$$W = -m \log \left(z^4 - a^4\right) + m \log \left(z^4 - b^4\right)$$

$$= -m \log \left(r^4 \cos 4\theta - a^4 + ir^4 \sin 4\theta\right) + m \log \left(r^4 \cos 4\Theta - b^4 + ir^4 \sin 4\Theta\right)$$

$$\psi = -m \left[ \tan^{-1} \frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - a^4} - \tan^{-1} \frac{r^4 \sin 4\Theta}{r^4 \cos 4\Theta - b^4} \right].$$

Thus the stream function of two-dimensional motion due to a source of strength $m$ at $(a,0)$ and an equal magnitude of sink at $(b,0)$ is

$$-m \tan^{-1} \frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 + b^4) \cos 4\theta + a^4 b^4}.$$
The velocity at the point $(r, \theta)$ in two-dimensional liquid motion due to a source and sink is

$$q = \frac{dW}{dz} = -\frac{4mr^3(a^4 - b^4)}{(r^8 - 2a^4r^4 \cos 4\theta + a^8)(r^8 - 2b^4r^4 \cos 4\theta + b^8)^{\frac{1}{2}}}.$$ 


4. Flow due to a Source and Sink at the Corners of Infinite Rectangle

Consider the infinite rectangle in the $z$-plane for which $0 \leq y \leq \pi$, $x \geq 0$. Use the transformation $t = \cosh z$, where $t = \xi + i\eta$ and $z = x + iy$. Therefore, $\xi = \cosh x \cos y$ and $\eta = \sinh x \sin y$, where $0 \leq y \leq \pi$, $x \geq 0$. If $y = 0$ and $0 < x < \infty$, then $1 < \xi < \infty$. If $x = 0$ and $0 \leq y \leq \pi$, then $-1 < \xi < 1$. If $y = \pi$ and $0 < x < \infty$, then $-\infty < \xi < -1$. Thus, the infinite rectangle in the $z$-plane for which $0 \leq y \leq \pi$, $x \geq 0$ into the half of the $t$-plane for which $\eta$ is positive.

Consider the two-dimensional irrotational motion of a liquid due to within the above infinite rectangle with a source and sink are placed at the corners $(0, 0)$ and $(0, \pi)$.

The source transforms into an equal source at $(1, 0)$ and the sink transforms into an equal sink at $(-1, 0)$. The complex potential for this motion is

$$W = -m \log(t - 1) + m \log(t + 1)$$

$$= -m \log \frac{\cosh z - 1}{\cosh z + 1} = -2m \log \tanh \frac{z}{2}$$

$$= -\lambda \log \tanh \frac{z}{2}, \text{ where } -2m = -\lambda.$$ 

$$\frac{dW}{dz} = -\frac{\lambda}{2} \frac{\sec h^2 \frac{z}{2}}{\tanh \frac{z}{2}} = -\frac{\lambda}{2 \sinh z}.$$
For the curve of equal pressure in the liquid, the velocity must be constant. Therefore,

\[ q^2 = \left( -\frac{\lambda}{\sinh z} \right)^2 = C_1^2, \]

where \( C_1 \) is a constant.

\[ \sinh z \sinh z = C_2^2 \]

\[ \cosh (z + \bar{z}) - \cosh (z - \bar{z}) = C_2^2. \]

Therefore, the curves of equal pressure in the liquid are given by \( \sinh^2 x + \sin^2 y = C^2 \).

Acknowledgements

I would like to express my heartfelt gratitude to Dr Maung Maung Naing, Rector, Dr Si Si Khin, Pro-Rector and Dr Tint Moe Thuazar, Pro-Rector, Yadanabon University for their permission to carry out the research and their encouragement. And then, I am deeply indebted to Dr Mya Oo, Director(Retd), Department of Higher Education, Upper Myanmar for his valuable advices and guidance on this research paper.

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